

Exact partition functions for the Ω -deformed $\mathcal{N} = 2^*$ $SU(2)$ gauge theory

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ABSTRACT: We study the low energy effective action of the Ω -deformed $\mathcal{N} = 2^*$ $SU(2)$ gauge theory. It depends on the deformation parameters ϵ_1, ϵ_2 , the scalar field expectation value a , and the hypermultiplet mass m . We explore the plane $(\frac{m}{\epsilon_1}, \frac{\epsilon_2}{\epsilon_1})$ looking for special features in the multi-instanton contributions to the prepotential, motivated by what happens in the Nekrasov-Shatashvili limit $\epsilon_2 \rightarrow 0$. We propose a simple condition on the structure of poles of the k -instanton prepotential and show that it is admissible at a finite set of points in the above plane. At these special points, the prepotential has poles at fixed positions independent on the instanton number. Besides and remarkably, both the instanton partition function and the full prepotential, including the perturbative contribution, may be given in closed form as functions of the scalar expectation value a and the modular parameter q appearing in special combinations of Eisenstein series and Dedekind η function. As a byproduct, the modular anomaly equation can be tested at all orders at these points. We discuss these special features from the point of view of the AGT correspondence and provide explicit toroidal 1-blocks in non-trivial closed form. The full list of solutions with 1, 2, 3, and 4 poles is determined and described in details.

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1 Introduction and results

In this paper we consider the Ω -deformed $\mathcal{N} = 2^* SU(2)$ gauge theory in four dimensions and present novel closed expressions for its low energy effective action at special values of the deformation parameters. On general grounds, before deformation, the effective action of $\mathcal{N} = 2$ theories is computed by the Seiberg-Witten (SW) curve [1, 2]. It is the sum of a 1-loop perturbative correction and an infinite series of non-perturbative instantonic contributions that are weighted by the instanton counting parameter $q = e^{i\pi\tau}$ where τ is the complexified gauge coupling constant at low energy. Due to $\mathcal{N} = 2$ supersymmetry, the full effective action may be expressed in terms of the analytic prepotential $\mathcal{F}(a, m)$ depending on the vacuum expectation value a of the scalar in the adjoint gauge multiplet and on the mass m of the adjoint hypermultiplet [3].

Instead of applying the SW machinery, one may compute the effective action by topological twisting the theory and exploiting localization on the many-instanton moduli spaces [4–6]. Technically, this is made feasible by introducing the so-called Ω -deformation of the theory, *i.e.* a modification breaking 4d Poincaré invariance and depending on two parameters ϵ_1, ϵ_2 such that the initial theory is recovered when $\epsilon_1, \epsilon_2 \rightarrow 0$. The role of

the Ω -deformation is that of a complete regulator for the instanton moduli space integration [7–14]. In this approach, it is natural to introduce a well defined partition function $Z^{\text{inst}}(\epsilon_1, \epsilon_2, a, m)$ and its associated non-perturbative ϵ -deformed prepotential by means of

$$F^{\text{inst}}(\epsilon_1, \epsilon_2, a, m) = -\epsilon_1 \epsilon_2 \log Z^{\text{inst}}(\epsilon_1, \epsilon_2, a, m). \quad (1.1)$$

It is well established that the quantity in (1.1) is interesting at finite values of the deformation parameters ϵ_1, ϵ_2 , *i.e.* taking seriously the deformed theory. This is because the amplitudes appearing in the expansion $F^{\text{pert}} + F^{\text{inst}} = \sum_{n,g=0}^{\infty} F^{(n,g)} (\epsilon_1 + \epsilon_2)^{2n} (\epsilon_1 \epsilon_2)^g$ are related to the genus g partition function of the $\mathcal{N} = 2$ topological string [15–21] and satisfy a powerful holomorphic anomaly equation [22–25]. Actually, understanding the exact dependence on the deformation parameters is an interesting topic if one wants to resum the above expansion in higher genus amplitudes. Clearly, this issue is closely related to the Alday-Gaiotto-Tachikawa (AGT) correspondence [26] mapping deformed $\mathcal{N} = 2$ instanton partition functions to conformal blocks of a suitable CFT with assigned world-sheet genus and operator insertions. AGT correspondence may be checked by working order by order in the number of instantons [27–29]. For the $\mathcal{N} = 2^*$ ϵ -deformed $SU(2)$ gauge theory the relevant CFT quantity is the one-point conformal block on the torus, a deceptively simple object of great interest [30, 27, 31, 28, 32–37].

The AGT interpretation emphasizes the importance of modular properties in the deformed gauge theory. Indeed, it is known that SW methods can be extended to the case of non-vanishing deformation parameters ϵ_1, ϵ_2 [38, 39] and modular properties have been clarified in the undeformed case [40, 41] as well as in presence of the deformation [42, 43]. The major outcome of these studies are explicit resummations of the instanton expansion order by order in the large a regime. The coefficients of the $1/a$ powers are expressed in terms of quasi-modular functions of the torus *nome* q . This approach can be pursued in the gauge theory [44–50], in CFT language by AGT correspondence [34, 35, 51, 36], and also in the framework of the semiclassical WKB analysis [52, 53, 52, 54–57].

An important simpler setup where these problems may be addressed is the so-called Nekrasov-Shatashvili (NS) limit [58] where one of the two ϵ parameters vanishes. In this case, the deformed theory has an unbroken two dimensional $\mathcal{N} = 2$ super-Poincaré invariance and its supersymmetric vacua are related to the eigenstates of a quantum integrable system. Under this Bethe/gauge map, the non-zero deformation parameter ϵ plays the role of \hbar in the quantization of a classically integrable system. Saddle point methods allow to derive a deformed SW curve [59, 60] that can also be analyzed by matrix model methods [33, 52, 53, 61, 62]. In the specific case of the $\mathcal{N} = 2^*$ theory, the relevant integrable system is the elliptic Calogero-Moser system [58] and the associated spectral problem reduces to the study of the celebrated Lamé equation. Besides, if the hypermultiplet mass m is taken to be proportional to ϵ with definite special ratios $\frac{m}{\epsilon} = n + \frac{1}{2}$, where $n \in \mathbb{N}$, the spectral problem is n -gap. Remarkable simplifications occur in the k -instanton prepotential contributions F_k^{inst} [63] that may be obtained by expanding the eigenvalues of a Lamé equation in terms of its Floquet exponent. As a byproduct of this approach, it is possible to clarify the meaning of the poles that appear in the k -instanton prepotential at

special values $a = \mathcal{O}(\epsilon)$ of the vacuum expectation value a . Indeed, the pole singularities turn out to be an artifact of the instanton expansion.

In this paper, we inquire into similar problems when both the deformation parameters are switched on, *i.e.* by going beyond the Nekrasov-Shatashvili limit. In particular, we explore the (α, β) plane where α, β are real parameters entering the scaling relation

$$m = \alpha \epsilon_1, \quad \epsilon_2 = \beta \epsilon_1. \quad (1.2)$$

In other words, we keep the hypermultiplet mass to be proportional to one deformation parameter with ratio α , but ϵ_1, ϵ_2 are generic (β is just a convenient replacement of ϵ_2). By dimensional scaling, the prepotential is a function $\tilde{F}_{(\alpha, \beta)}(\nu)$ of the combination $\nu = 2a/\epsilon_1$ at the fixed point (α, β) .¹ The dependence on q is not written explicitly. After this stage preparation, the claim of this paper is the following

There exists a finite set of N -poles points (α, β) such that the k -instanton prepotential is a rational function of ν with poles at a fixed set of positions $\nu \in \{\nu_1, \dots, \nu_N\}$ independent on k .

This claim is motivated by our previous analysis in the restricted Nekrasov-Shatashvili limit [63] and is far from obvious. Most important, it has far reaching consequences. At the special N -poles points, we show that the instanton partition function and the perturbative part of the prepotential take the exact form

$$\begin{aligned} \tilde{Z}_{(\alpha, \beta)}^{\text{inst}}(\nu) &= \frac{\nu^{2N} + \sum_{n=1}^N \nu^{2(N-n)} \mathcal{M}_{2n}(q)}{(\nu^2 - \nu_1^2) \dots (\nu^2 - \nu_N^2)} [q^{-\frac{1}{12}} \eta(\tau)]^{2(h_m-1)}, \\ \tilde{F}_{(\alpha, \beta)}^{\text{pert}}(\nu) &= -\beta h_m \log \frac{\nu}{\Lambda} - \beta \log \prod_{n=1}^N \left(1 - \frac{\nu_n^2}{\nu^2}\right), \quad h_m = \frac{(\beta+1)^2 - 4\alpha^2}{4\beta}, \end{aligned} \quad (1.3)$$

where \mathcal{M}_{2n} is a polynomial in the Eisenstein series E_2, E_4, E_6 with total modular degree $2n$ with coefficients depending on α, β , and $h_m \in \mathbb{N}$. The total prepotential is thus remarkably simple and reads

$$\tilde{F}_{(\alpha, \beta)}(\nu) = -\beta h_m \log \frac{\nu}{\Lambda} - \beta \log \left(1 + \sum_{n=1}^N \frac{\mathcal{M}_{2n}(q)}{\nu^{2n}}\right). \quad (1.4)$$

These explicit expression satisfy the modular anomaly equation expressing S-duality discussed in [44–49]. By applying the AGT dictionary, (1.3) and (1.4) predict toroidal blocks in closed form at very specific values of the central charge c and of the inserted operator conformal dimension h_m – the perturbative part providing interesting special instances of the 3-point DOZZ Liouville correlation function. These results are derived and tested by giving a complete list of all the $N \leq 4$ poles points. These turns out to be 4, 7, 12, and 11 at $N = 1, 2, 3, 4$ respectively.

¹ We shall systematically add a tilde to quantities that are considered under (1.2) and expressed in terms of the variable ν .

The plan of the paper is the following. In Sec. (2) we determine the 1-pole points by a direct inspection of the instanton prepotential contributions. In Sec. (2.1) we discuss the special features of the instanton partition function at the 1-pole points. The AGT interpretation is analyzed in Sec. (3) where we also provide various explicit CFT tests of the proposed partition functions. In Sec. (4) we discuss the perturbative part of the prepotential at the 1-pole points. In Sec. (5) the analysis is extended to N -poles points and the cases $N = 2, 3$ are fully classified. Finally, Sec. (6) presents a list of special toroidal blocks. Various appendices are devoted to additional comments.

2 Looking for simplicity beyond the Nekrasov-Shatashvili limit

As discussed in the Introduction, we are interested in the scaling limit (1.2). The instanton partition function is $Z^{\text{inst}} = Z^{\text{inst}}(\epsilon_1, \epsilon_2, a, m)$ and it is convenient to introduce

$$\tilde{Z}_{(\alpha, \beta)}^{\text{inst}}(\nu) = Z^{\text{inst}}\left(\epsilon_1, \beta \epsilon_1, \frac{\epsilon_1 \nu}{2}, \alpha \epsilon_1\right) = Z^{\text{inst}}\left(1, \beta, \frac{\nu}{2}, \alpha\right), \quad (2.1)$$

where we used dimensional scaling independence to remove ϵ_1 . Similarly, we define

$$F^{\text{inst}} = -\epsilon_1 \epsilon_2 \log Z^{\text{inst}}, \quad \tilde{F}_{(\alpha, \beta)}^{\text{inst}}(\nu) = -\beta \log \tilde{Z}_{(\alpha, \beta)}^{\text{inst}}(\nu). \quad (2.2)$$

We shall omit the explicit (α, β) index when obvious. Besides, the partition function is even in α and we shall always consider $\alpha > 0$.

According to the claim presented in the Introduction, we now look for special points (α, β) such that the k -instanton Nekrasov function takes the form

$$\tilde{F}_k^{\text{inst}}(\nu) = \frac{P_k(\nu)}{(\nu^2 - \nu_1^2)^k}, \quad (2.3)$$

with a polynomial $P_k(\nu)$ and a single pole $\nu_1 \geq 0$ in the variable $|\nu|$. The Ansatz (2.3) is a non-trivial requirement. It is motivated by the analysis in [63], but its admissibility is actually one of the results of our investigation. To explore the constraints that (2.3) imposes, we begin by looking at the simple one-instanton case. ² For $k = 1$ we have the explicit expression

$$\tilde{F}_1^{\text{inst}}(\nu) = -\frac{(2\alpha - \beta + 1)(2\alpha + \beta - 1)(4\alpha^2 + 3\beta^2 + 6\beta - 4\nu^2 + 3)}{8(\beta - \nu + 1)(\beta + \nu + 1)}, \quad (2.4)$$

and there is a simple pole $\nu_1 = |\beta + 1|$. At the two-instanton level, $k = 2$, the denominator of $\tilde{F}_2(\nu)$ turns out to vanish at

$$|\nu| = \beta + 1(\text{order } 2), \beta + 2, 2\beta + 1. \quad (2.5)$$

² The functions $\tilde{F}_k^{\text{inst}}$ may be computed by the beautiful Nekrasov formula [5]. Alternatively, for a gauge algebra $\mathfrak{g} \in \{A_r, B_r, C_r, D_r\}$ one can also apply the methods described in [4, 5, 8, 64, 65, 11, 66].

Special cases occur when one of the poles coincides with those at $\nu_1 = |\beta + 1|$. This happens for

$$\beta = -1, -\frac{3}{2}, -\frac{2}{3}. \quad (2.6)$$

These values must be analyzed separately. Looking at higher values of k we identify the only non-trivial cases consistent with (2.3)³

$$(\alpha, \beta) = \left(\frac{7}{4}, -\frac{3}{2}\right), \left(\frac{7}{6}, -\frac{2}{3}\right). \quad (2.7)$$

Finally, if β is not in the set (2.6), one checks that \tilde{F}_2 takes the form (2.3) if

$$(\alpha, \beta) = \left(\frac{5}{2}, -2\right), \left(\frac{5}{4}, -\frac{1}{2}\right). \quad (2.8)$$

Pushing the calculation up to 12 instantons, we confirm that the points in (2.7) and (2.8) agree with the Ansatz (2.3). Thus, the 1-pole condition (2.3) selects the following distinct 4 special points

$$X_1 = \left(\frac{5}{2}, -2\right), \quad X_2 = \left(\frac{7}{4}, -\frac{3}{2}\right), \quad X_3 = \left(\frac{7}{6}, -\frac{2}{3}\right), \quad X_4 = \left(\frac{5}{4}, -\frac{1}{2}\right). \quad (2.9)$$

2.1 Back to the instanton partition functions

We could analyze further the structure of the prepotential in (2.3) at the special points X_i in (2.9) by looking for regularities in the polynomials $P_k(\nu)$. However, it is much more convenient to go back to the instanton partition function. To see why, let us consider X_1 as a first illustration. We find indeed the simple expansion

$$\begin{aligned} \tilde{Z}_{X_1}^{\text{inst}}(\nu) = & 1 - \frac{4(\nu^2 - 7)q^2}{\nu^2 - 1} + \frac{2(\nu^2 - 13)q^4}{\nu^2 - 1} + \frac{8(\nu^2 - 19)q^6}{\nu^2 - 1} - \frac{5(\nu^2 - 25)q^8}{\nu^2 - 1} \\ & - \frac{4(\nu^2 - 31)q^{10}}{\nu^2 - 1} - \frac{10(\nu^2 - 37)q^{12}}{\nu^2 - 1} + \frac{8(\nu^2 - 43)q^{14}}{\nu^2 - 1} + \frac{9(\nu^2 - 49)q^{16}}{\nu^2 - 1} \\ & + \frac{14(\nu^2 - 61)q^{20}}{\nu^2 - 1} + \mathcal{O}(q^{22}). \end{aligned} \quad (2.10)$$

After some educated trial and error, we recognize that (2.10) is the expansion of the following expression

$$\tilde{Z}_{X_1}^{\text{inst}}(\nu) = \frac{\nu^2 - E_2(q)}{\nu^2 - 1} q^{-\frac{1}{3}} \eta(\tau)^4, \quad (2.11)$$

where

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{k=1}^{\infty} (1 - q^{2k}), \quad q = e^{i\pi\tau}, \quad (2.12)$$

³Here, trivial means a constant $\tilde{F}_k(\nu)$.

and E_2 is an Eisenstein series.⁴ Similar expressions are found at the other three special points. The detailed formulas are

$$\begin{aligned}\tilde{Z}_{X_2}^{\text{inst}}(\nu) &= \frac{4\nu^2 - E_2(q)}{4\nu^2 - 1} q^{-\frac{1}{6}} \eta(\tau)^2, \\ \tilde{Z}_{X_3}^{\text{inst}}(\nu) &= \frac{9\nu^2 - E_2(q)}{9\nu^2 - 1} q^{-\frac{1}{6}} \eta(\tau)^2, \\ \tilde{Z}_{X_4}^{\text{inst}}(\nu) &= \frac{4\nu^2 - E_2(q)}{4\nu^2 - 1} q^{-\frac{1}{3}} \eta(\tau)^4.\end{aligned}\tag{2.13}$$

The associated all-instanton Nekrasov functions are

$$\begin{aligned}\tilde{F}_{X_1}^{\text{inst}}(\nu) &= 8 \log[q^{-\frac{1}{12}} \eta(\tau)] + 2 \log\left(\frac{\nu^2 - E_2}{\nu^2 - 1}\right), \\ \tilde{F}_{X_2}^{\text{inst}}(\nu) &= 3 \log[q^{-\frac{1}{12}} \eta(\tau)] + \frac{3}{2} \log\left(\frac{4\nu^2 - E_2}{4\nu^2 - 1}\right), \\ \tilde{F}_{X_3}^{\text{inst}}(\nu) &= \frac{4}{3} \log[q^{-\frac{1}{12}} \eta(\tau)] + \frac{2}{3} \log\left(\frac{9\nu^2 - E_2}{9\nu^2 - 1}\right), \\ \tilde{F}_{X_4}^{\text{inst}}(\nu) &= 2 \log[q^{-\frac{1}{12}} \eta(\tau)] + \frac{1}{2} \log\left(\frac{4\nu^2 - E_2}{4\nu^2 - 1}\right).\end{aligned}\tag{2.14}$$

Equations (2.13) and (2.14) are already remarkable because they are non-trivial closed expressions for the instanton partition function, or prepotential, at all instanton numbers. It is clear that it would be nice to provide some clarifying interpretation for this features at the special points X_i . In the next section, we shall examine the clues coming from AGT correspondence.

3 AGT interpretation

According to the AGT correspondence, the instanton partition function of $\mathcal{N} = 2^* SU(2)$ gauge theory is [26, 28, 27]

$$Z^{\text{inst}}(q, a, m) = \left[\prod_{k=1}^{\infty} (1 - q^{2k}) \right]^{-1+2h_m} \mathcal{F}_{h_m}^h(q),\tag{3.1}$$

where $\mathcal{F}_{\alpha}^m(q)$ is the 1-point toroidal block of the Virasoro algebra of central charge $c = 1 + 6Q^2$ on a torus whose modulus is q , with one operator of dimension h_m inserted and a primary of dimension h in the intermediate channel. The precise dictionary in terms of the deformation parameters is

$$\begin{aligned}b &= \sqrt{\epsilon_2/\epsilon_1}, & Q &= b + b^{-1}, \\ h_m &= \frac{Q^2}{4} - \frac{m^2}{\epsilon_1 \epsilon_2}, & h &= \frac{Q^2}{4} - \frac{a^2}{\epsilon_1 \epsilon_2}.\end{aligned}\tag{3.2}$$

⁴ Our convention is

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}, \quad E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}.$$

Modular properties of these quantities may be found, for instance, in [67].

Assuming the scaling relations (1.2), the expressions in (3.2) read

$$\begin{aligned} b &= \sqrt{\beta}, & Q &= \beta^{\frac{1}{2}} + \beta^{-\frac{1}{2}}, \\ h_m &= \frac{(\beta+1)^2 - 4\alpha^2}{4\beta}, & h &= \frac{(\beta+1)^2 \epsilon_1^2 - 4a^2}{4\beta \epsilon_1^2}, \end{aligned} \quad (3.3)$$

with central charge

$$c = 13 + 6 \left(\beta + \frac{1}{\beta} \right). \quad (3.4)$$

In particular, at the four points X_i we obtain the following values for (c, h_m)

	X_1	X_2	X_3	X_4
c	-2	0	0	-2
h_m	3	2	2	3

(3.5)

Of course, points appear in pairs with the same central charge and β values related by $\beta \rightarrow \beta^{-1}$. More remarkably, the associated values of the parameter α is always such that h_m is a positive integer. The toroidal block has a universal prefactor $q^{\frac{1}{12}}/\eta(\tau)$ that is its value at $h \rightarrow \infty$. Comparing (3.1) with (2.13) we can write the general form for all four X_i points as

$$\begin{aligned} \tilde{Z}_{(\alpha, \beta)}^{\text{inst}}(v) &= \frac{v^2 - v_1^2 E_2(q)}{v^2 - v_1^2} [q^{-\frac{1}{12}} \eta(\tau)]^{2(h_m-1)}, & v_1 &= |\beta + 1|, \\ \tilde{F}_{(\alpha, \beta)}^{\text{inst}}(v) &= -2\beta(h_m - 1) \log[q^{-\frac{1}{12}} \eta(\tau)] - \beta \log \frac{v^2 - v_1^2 E_2}{v^2 - v_1^2}. \end{aligned} \quad (3.6)$$

Correspondingly, the net prediction for the toroidal block at the above central charge and insertion dimension is

$$\mathcal{F}_{h_m}^h(q, c) = \frac{q^{\frac{1}{12}}}{\eta(\tau)} \left[1 + \frac{c-1}{24h} (E_2(q) - 1) \right], \quad (c, h_m) = (0, 2) \text{ or } (-2, 3). \quad (3.7)$$

We remark that the above (c, h_m) may well be pathological for a physical CFT. Nevertheless, the toroidal block is defined by the Virasoro algebra for arbitrary values of c, h_m , and h . Eq. (3.7) must be taken in this sense. We checked (3.7) against Zamolodchikov recursive determination of the toroidal block [68–70] with perfect agreement. Of course, by AGT, this is same as Nekrasov calculation. The remarkably simple form (3.7) is clearly consistent with general results for the torus block. For instance, at leading and next-to-leading order and generic operator dimensions we have [35]

$$\mathcal{F}_{h_m}^h(c, q) = 1 + \mathcal{F}_1(h, h_m, c) q^2 + \mathcal{F}_2(h, h_m, c) q^4 + \dots, \quad (3.8)$$

where

$$\begin{aligned} \mathcal{F}_1(h, h_m, c) &= 1 + \frac{h_m(h_m - 1)}{2h}, \\ \mathcal{F}_2(h, h_m, c) &= [4h(2ch + c + 16h^2 - 10h)]^{-1} \end{aligned} \quad (3.9)$$

$$\begin{aligned} & \left[(8ch + 3c + 128h^2 + 56h) h_m^2 + (-8ch - 2c - 128h^2) h_m \right. \\ & \left. + (c + 8h) h_m^4 + (-2c - 64h) h_m^3 + 16ch^2 + 8ch + 128h^3 - 80h^2 \right]. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{F}_{h_m=2}^h(c=0, q) &= 1 + \left(1 + \frac{1}{h}\right) q^2 + \left(2 + \frac{4}{h}\right) q^4 + \dots, \\ \mathcal{F}_{h_m=3}^h(c=-2, q) &= 1 + \left(1 + \frac{3}{h}\right) q^2 + \left(2 + \frac{12}{h}\right) q^4 + \dots, \end{aligned} \quad (3.10)$$

in full agreement with (3.7) for $c = 0, -2$. Notice also that (3.7) may be written

$$\frac{q^{\frac{1}{12}}}{\eta(\tau)} \left[1 + \frac{c-1}{24h} (E_2(q) - 1) \right] = \left(1 + \frac{1-c}{2h} q \partial q \right) \frac{q^{\frac{1}{12}}}{\eta(\tau)} = \sum_{k=0}^{\infty} \left(1 + \frac{1-c}{h} k \right) P_k q^{2k}, \quad (3.11)$$

where P_k are the coefficients of the expansion of $\prod_{k=1}^{\infty} (1 - q^{2k})^{-1}$, i.e. the number of unrestricted partitions of k (n^m means m copies of n)

$$\begin{aligned} P_1 &= \#\{(1)\} = 1, \quad P_2 = \#\{(2), (1^2)\} = 2, \quad P_3 = \#\{(3), (2, 1), (1^3)\} = 3, \\ P_4 &= \#\{(4), (3, 1), (2^2), (2, 1^2), (1^4)\} = 5, \\ P_5 &= \#\{(5), (4, 1), (3, 2), (3, 1^2), (2^2, 1), (2, 1^3), (1^5)\} = 7, \text{ and so on.} \end{aligned} \quad (3.12)$$

3.1 Explicit CFT computations

The $(c, h_m) = (0, 2)$ conformal block

It is instructive to derive the result (3.7) at $c = 0$ from a direct CFT calculation.⁵ In other words, we want to show that

$$\mathcal{F}_{h_m=2}^h(q, c=0) = \frac{q^{\frac{1}{12}}}{\eta(\tau)} \left[1 - \frac{1}{24h} (E_2(q) - 1) \right] = \sum_{k=0}^{\infty} \left(1 + \frac{1}{h} k \right) P_k q^{2k}, \quad (3.13)$$

where we used (3.11). The toroidal block is obtained as

$$\mathcal{F}_{h_m}^h(q, c) = q^{-h + \frac{c}{12}} \text{Tr}_h \left(q^{L_0 - \frac{c}{12}} \varphi_{h_m}(1) \right), \quad (3.14)$$

where the trace is over the descendants of φ_h . The starting point is thus conformal descendant decomposition of the *diagonal* part of the OPE

$$\begin{aligned} \varphi_{h_m}(x) \varphi_h(0) &= \sum_Y x^{-h_m + |Y|} \beta^Y L_{-Y} \varphi_h(0) \\ &= x^{-h_m} (1 + x \beta^{(1)} L_{-1} + x^2 (\beta^{(2)} L_{-2} + \beta^{(1,1)} L_{-1}^2) + \dots) \varphi_h(0), \end{aligned} \quad (3.15)$$

⁵ CFT at $c = 0$ is obviously quite special since the 2-point function of the stress energy tensor is then zero, so strictly speaking, the theory is not conformal any more (since the stress-tensor vanishes identically). However, as we remarked, we are considering the toroidal block as a well-defined function of (c, h_m, h) that may be regarded as the solution to the Zamolodchikov recursion relations. It would be interesting to revisit our calculation in the language of logarithmic CFT, see for instance [71, 72].

where Y denotes a unrestricted partition of $|Y|$ and L_n are Virasoro generators

$$\begin{aligned} Y &= \{k_1 \geq k_2 \geq \dots > 0\}, & |Y| &= k_1 + k_2 + \dots, \\ L_{-Y} &= L_{-k_1} L_{-k_2} \dots \end{aligned} \quad (3.16)$$

The coefficients β in (3.15) are determined by conformal symmetry and are functions of h, h_m, c . As a consequence of unbroken conformal symmetry ($c = 0$) and of the fact that $h_m = 2$ is the same dimension as that of the energy momentum tensor, one finds that the only vanishing β coefficients are those associated with simple L_{-n} descendants. Besides they are all equal

$$\beta^{(n)} = \frac{1}{h}, \quad \beta^{(n,n')} = 0, \quad \beta^{(n,n',n'')} = 0, \quad \dots \quad (3.17)$$

Just to give an example, the explicit β coefficients at level 2 are

$$\begin{aligned} \beta^{(1)} &= \frac{h_m}{2h}, & \beta^{(2)} &= \frac{(1 + 8h - 3h_m)h_m}{c(1 + 2h) + 2h(8h - 5)}, \\ \beta^{(1,1)} &= \frac{h_m(c - 16h + (c + 8h)h_m)}{4h(c(1 + 2h) + 2h(8h - 5))}. \end{aligned} \quad (3.18)$$

Computing them at $(c, h_m) = (0, 2)$ we see that indeed

$$\beta^{(1)} = \beta^{(2)} = \frac{1}{h}, \quad \beta^{(1,1)} = 0. \quad (3.19)$$

Hence, if we apply (3.15) to the vacuum, we get ⁶

$$\varphi_2(x) |h\rangle = \frac{1}{h} \sum_{n=0}^{\infty} x^{n-2} L_{-n} |h\rangle. \quad (3.20)$$

Now, to get the torus block, we need to evaluate the diagonal matrix elements of $\varphi_2(x)$. Using

$$[L_n, \varphi_2(x)] = x^n (x \partial + 2(n+1)) \varphi_2(x), \quad (3.21)$$

we obtain with one index

$$\begin{aligned} \varphi_2(x) L_{-k} |h\rangle &= -[L_{-k}, \varphi_2(x)] |h\rangle + L_{-k} \varphi_2(x) |h\rangle \\ &= \frac{1}{h} \sum_{n=0}^{\infty} (2k - n) x^{n-k-2} L_{-n} |h\rangle + \frac{1}{h} \sum_{n=0}^{\infty} x^{n-2} L_{-k} L_{-n} |h\rangle \\ &= \dots + \frac{1}{x^2} \left(1 + \frac{k}{h}\right) L_{-k} |h\rangle + \dots, \end{aligned} \quad (3.22)$$

where we have shown only the diagonal entry. Adding one index each time, a similar calculation shows that for any number of indices

$$\varphi_2(x) L_{-Y} |h\rangle = \dots + \frac{1}{x^2} \left(1 + \frac{|Y|}{h}\right) L_{-Y} |h\rangle + \dots \quad (3.23)$$

Thus the diagonal matrix element of $\varphi_2(x)$ associated with the Y descendent depends only on $|Y|$. The number of Y with fixed $|Y|$ is the number $P_{|Y|}$ of unrestricted partitions of $|Y|$. Summing over Y with $|Y| = k$ we prove (3.13).

⁶Notice that $h_m = 2$ is not enough to achieve such simplification. A vanishing central charge is also needed to remove descendants with multiple applications of Virasoro operators.

The $(c, h_m) = (-2, 3)$ conformal block

A similar computation for $c = -2$ and $h_m = 3$ is apparently quite less trivial. The main reason is that the β coefficients in the conformal decomposition (3.15) do not trivialize in this case. This complication forbids us to prove the wanted result in general. Nevertheless, we provide an explicit check at level 4. Of course, one could simply use the recursive definition of the toroidal block, but our brute force calculation is perhaps more transparent. Besides, it emphasizes the difference compared with the previous $(c, h_m) = (0, 2)$ case. The starting point is again the OPE (3.15) that now takes the following form up to level 4 descendants

$$\begin{aligned} \varphi_3(x) \varphi_h(0) = & x^{-3} (1 + x \beta^{(1)} L_{-1} + x^2 (\beta^{(2)} L_{-2} + \beta^{(1,1)} L_{-1}^2) \\ & + x^3 (\beta^{(3)} L_{-3} + \beta^{(2,1)} L_{-2} L_{-1} + \beta^{(1,1,1)} L_{-1}^3) \\ & + x^4 (\beta^{(4)} L_{-4} + \beta^{(3,1)} L_{-3} L_{-1} + \beta^{(2,2)} L_{-2}^2 + \beta^{(2,1,1)} L_{-2} L_{-1}^2 + \beta^{(1,1,1,1)} L_{-1}^4) + \dots) \varphi_h(0), \end{aligned} \quad (3.24)$$

with the simple but non trivial β coefficients

$$\begin{aligned} \beta^{(1)} &= \frac{3}{2h}, \quad \beta^{(2)} = \frac{12}{8h+1}, \quad \beta^{(1,1)} = \frac{3}{h(8h+1)}, \quad \beta^{(3)} = \frac{24h-1}{2h(8h+1)}, \\ \beta^{(2,1)} &= \frac{6h+1}{h^2(8h+1)}, \quad \beta^{(1,1,1)} = -\frac{1}{2h^2(8h+1)}, \quad \beta^{(4)} = \frac{3(32h^2-20h+1)}{h(8h-3)(8h+1)}, \\ \beta^{(3,1)} &= \frac{3(16h^2-2h-1)}{h^2(8h-3)(8h+1)}, \quad \beta^{(2,2)} = \frac{24}{(8h-3)(8h+1)}, \\ \beta^{(2,1,1)} &= -\frac{3(4h+1)}{h^2(8h-3)(8h+1)}, \quad \beta^{(1,1,1,1)} = \frac{3}{2h^2(8h-3)(8h+1)} \end{aligned}$$

As in (3.23), we write

$$\varphi_3(x) L_{-Y}|h\rangle = \dots + \frac{1}{x^3} M_Y L_{-Y}|h\rangle + \dots, \quad (3.25)$$

for certain coefficients M_Y functions of h . At level 1, we have only

$$M_1 = 1 + \frac{3}{h}. \quad (3.26)$$

At level 2,

$$\begin{aligned} M_2 &= \frac{8h+49}{8h+1}, \quad M_{1,1} = \frac{8h^2+49h+12}{h(8h+1)}, \\ \sum_{|Y|=2} M_Y &= 2 + \frac{12}{h}. \end{aligned} \quad (3.27)$$

At level 3,

$$\begin{aligned} M_3 &= \frac{8h^2+73h-3}{h(8h+1)}, \quad M_{2,1} = \frac{8h^2+73h+3}{h(8h+1)}, \quad M_{1,1,1} = \frac{8h^2+73h+27}{h(8h+1)}, \\ \sum_{|Y|=3} M_Y &= 3 + \frac{27}{h}. \end{aligned} \quad (3.28)$$

At level 4

$$\begin{aligned}
M_4 &= \frac{64h^3 + 752h^2 - 483h + 24}{h(8h-3)(8h+1)}, & M_{3,1} &= \frac{64h^3 + 752h^2 - 291h - 24}{h(8h-3)(8h+1)}, \\
M_{2,2} &= \frac{64h^2 + 752h - 99}{(8h-3)(8h+1)}, & M_{2,1,1} &= \frac{h+12}{h}, & M_{1,1,1,1} &= \frac{8h^2 + 97h + 48}{h(8h+1)}, \\
\sum_{|Y|=4} M_Y &= 5 + \frac{60}{h}.
\end{aligned} \tag{3.29}$$

Putting all together we agree with (3.11) at $c = -2$. It would be nice to prove the agreement at all levels, possibly working in a definite $c = -2$ CFT like the triplet model considered in [73, 74].

4 Perturbative part of the prepotential at the points X_i

The prepotential has also a perturbative part, related by AGT to the DOZZ 3-point function in the Liouville theory [75, 69, 76]. The general expression for the perturbative part of the prepotential is ($\tilde{m} = m + \frac{\epsilon_1 + \epsilon_2}{2}$) [4, 5]

$$F^{\text{pert}} = \epsilon_1 \epsilon_2 \left[\gamma_{\epsilon_1, \epsilon_2}(2a) + \gamma_{\epsilon_1, \epsilon_2}(-2a) - \gamma_{\epsilon_1, \epsilon_2}(2a + \tilde{m}) - \gamma_{\epsilon_1, \epsilon_2}(-2a + \tilde{m}) \right], \tag{4.1}$$

where

$$\gamma_{\epsilon_1, \epsilon_2}(x) = \frac{d}{ds} \left(\frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} \frac{t^s e^{-tx}}{(e^{-\epsilon_1 t} - 1)(e^{-\epsilon_2 t} - 1)} \right) \Big|_{s=0}, \tag{4.2}$$

and Λ is a renormalization scale. Evaluating F_{pert} by expanding at small ϵ_1 and resumming, we find that at all X_i points it is possible to write

$$F^{\text{pert}} = \frac{4\alpha^2 - (\beta + 1)^2}{4} \epsilon_1^2 \log \frac{2a}{\Lambda} - \beta \epsilon_1^2 \log \left[1 - (1 + \beta)^2 \frac{\epsilon_1^2}{4a^2} \right]. \tag{4.3}$$

Again, this appears to be a special feature of the X_i points because it is not possible to give such a simple expression for F^{pert} at generic ϵ_1, ϵ_2 from (4.1). With a redefinition of the UV cutoff, this may be written in the following suggestive form that we shall generalize later

$$\tilde{F}^{\text{pert}}(\nu) = -\beta h_m \log \frac{\nu}{\Lambda} - \beta \log \left(1 - \frac{\nu_1^2}{\nu^2} \right). \tag{4.4}$$

5 Full prepotential and generalization to N -poles points

If we combine the perturbative (4.4) and instanton (3.6) parts of the prepotential, we obtain the remarkably simple expression

$$\tilde{F} = \tilde{F}^{\text{pert}} + \tilde{F}^{\text{inst}} = -\beta h_m \log \frac{\nu}{\Lambda} - \beta \log \left(1 + \frac{\gamma E_2(q)}{\nu^2} \right), \tag{5.1}$$

with a certain coefficient γ . This suggests that it is convenient to organize the total prepotential in the form

$$\tilde{F} = -\beta h_m \log \frac{\nu}{\Lambda} - \beta \log \left(1 + \sum_{n=1}^{\infty} \frac{\mathcal{M}_{2n}(q)}{\nu^{2n}} \right), \quad (5.2)$$

where $M_{2n}(q)$ is a polynomial in $E_{2,4,6}$ of (quasi-) modular degree $2n$. We emphasize again that the Ansatz (5.2) is non trivial because \tilde{F}_{tot} is a combination of the perturbative and instanton contributions. Our claim is that (5.2) can be truncated at maximum degree $n = N$ for a special set of points (α, β) . At such points, the instanton partition function takes the special form, see (1.3)

$$\tilde{Z}_{(\alpha, \beta)}^{\text{inst}}(\nu) = \frac{\nu^{2N} + \sum_{n=1}^N \nu^{2(N-n)} \mathcal{M}_{2n}(q)}{(\nu^2 - \nu_1^2) \dots (\nu^2 - \nu_N^2)} [q^{-\frac{1}{12}} \eta(\tau)]^{2(h_m-1)}. \quad (5.3)$$

To see how this works in practice, let us parametrize

$$\begin{aligned} \mathcal{M}_2 &= \gamma_2 E_2, & \mathcal{M}_4 &= \gamma_4 E_4 + \gamma_{2,2} E_2^2, \\ \mathcal{M}_6 &= \gamma_6 E_6 + \gamma_{2,4} E_2 E_4 + \gamma_{2,2,2} E_2^3, \\ \mathcal{M}_8 &= \gamma_{2,6} E_2 E_6 + \gamma_{4,4} E_4^2 + \gamma_{2,2,4} E_2^2 E_4 + \gamma_{2,2,2,2} E_2^4. \end{aligned} \quad (5.4)$$

Replacing these combinations in (5.3) and comparing with the Nekrasov function at 5 instanton we obtain explicit expressions for the γ coefficients in terms of α, β . The first coefficients are

$$\begin{aligned} \gamma_4 &= -\frac{1}{23040\beta} (-2\alpha + \beta - 1)(-2\alpha + \beta + 1)(2\alpha + \beta - 1)(2\alpha + \beta + 1) \\ &\quad (-4\alpha^2 + 13\beta^2 + 20\beta + 13), \\ \gamma_{2,2} &= \frac{1}{73728\beta^2} (-2\alpha + \beta - 1)(-2\alpha + \beta + 1)(2\alpha + \beta - 1)(2\alpha + \beta + 1) \\ &\quad (-4\alpha^2 + \beta^2 - 10\beta + 1)(-4\alpha^2 + \beta^2 - 6\beta + 1). \end{aligned} \quad (5.5)$$

The non trivial solutions of $\gamma_4 = \gamma_{2,2} = 0$ (as always, up to the choice $\alpha > 0$) are

$$(\alpha, \beta) = \left(\frac{7}{6}, -\frac{2}{3} \right), \quad \left(\frac{5}{4}, -\frac{1}{2} \right), \quad \left(\frac{11}{6}, -\frac{2}{3} \right), \quad \left(\frac{5}{2}, -2 \right), \quad (5.6)$$

that is precisely the four points X_i . The coefficient γ_2 is also an output of the calculation and its general expression is

$$\gamma_2 = \frac{(-2\alpha + \beta - 1)(-2\alpha + \beta + 1)(2\alpha + \beta - 1)(2\alpha + \beta + 1)}{192\beta}. \quad (5.7)$$

Finally, comparing (5.2) and (5.3) we determine the perturbative part of the prepotential to be, see (1.3)

$$\tilde{F}_{(\alpha, \beta)}^{\text{pert}}(\nu) = -\beta h_m \log \frac{\nu}{\Lambda} - \beta \log \prod_{n=1}^N \left(1 - \frac{\nu_n^2}{\nu^2} \right), \quad (5.8)$$

that is a nice generalization of (4.4). In all cases, we have confirmed that this is in full agreement with the explicit evaluation of the general expression (4.1). We remark once again that it is remarkable that at the special n -poles points, it is possible to get such a simple expression.

Performing the above analysis for 2- and 3- poles Nekrasov functions, we find the following complete results. We have four 1- and seven 2-poles Nekrasov functions that are fully characterized by the following tables where we have separated by a horizontal line points with different (c, h_m)

(5.9)

α	β	c	h_m	γ_2	ν_1
$\frac{5}{2}$	-2	-2	3	-1	1
$\frac{5}{4}$	$-\frac{1}{2}$	-2	3	$-\frac{1}{4}$	$\frac{1}{2}$
$\frac{7}{4}$	$-\frac{3}{2}$	0	2	$-\frac{1}{4}$	$\frac{1}{2}$
$\frac{7}{6}$	$-\frac{2}{3}$	0	2	$-\frac{1}{9}$	$\frac{1}{3}$

α	β	c	h_m	γ_2	γ_4	$\gamma_{2,2}$	ν_1	ν_2
4	-3	-7	5	-5	$\frac{1}{4}$	$\frac{15}{4}$	1	2
$\frac{4}{3}$	$-\frac{1}{3}$	-7	5	$-\frac{5}{9}$	$\frac{1}{324}$	$\frac{5}{108}$	$\frac{1}{3}$	$\frac{2}{3}$
$\frac{13}{4}$	$-\frac{5}{2}$	$-\frac{22}{5}$	4	$-\frac{5}{2}$	$\frac{1}{24}$	$\frac{25}{48}$	$\frac{1}{2}$	$\frac{3}{2}$
$\frac{13}{10}$	$-\frac{2}{5}$	$-\frac{22}{5}$	4	$-\frac{2}{5}$	$\frac{2}{1875}$	$\frac{1}{75}$	$\frac{1}{5}$	$\frac{3}{5}$
2	-1	1	4	-1	$-\frac{1}{12}$	$\frac{1}{12}$	0	1
$\frac{11}{4}$	$-\frac{3}{2}$	0	5	$-\frac{5}{2}$	$-\frac{3}{8}$	$\frac{15}{16}$	$\frac{1}{2}$	$\frac{3}{2}$
$\frac{11}{6}$	$-\frac{2}{3}$	0	5	$-\frac{10}{9}$	$-\frac{2}{27}$	$\frac{5}{27}$	$\frac{1}{3}$	1

Looking for 3-poles points, we identify 12 cases whose full data is collected in the Table in (A.1). With 4-poles, we found 11 solutions collected in the two tables (A.2) and (A.3). Notice that in all presented cases, the parameter β is always rational negative. This implies that central charge takes the form of *extended* minimal models

$$c = 1 - 6 \frac{(p - q)^2}{pq}, \quad (5.10)$$

with coprime integers p, q . Here, the extension is due to the fact that the minimum value of p, q is one instead of 2, see also Fig. (1).

5.1 Constraints from the modular anomaly equation

An important test of the expression (5.2) is the validity of the modular anomaly equation expressing S-duality [44–49]. This is a non-trivial constraint capturing the dependence on the quasi-modular series E_2 and reads [45]

$$\frac{\partial \tilde{F}}{\partial E_2} + \frac{1}{12} \left(\frac{\partial \tilde{F}}{\partial \nu} \right)^2 - \frac{\beta}{12} \frac{\partial^2 \tilde{F}}{\partial \nu^2} = 0. \quad (5.11)$$

An alternative form is obtained by expanding at large ν and identifying the coefficients h_ℓ in

$$\tilde{F} = h_0 \log \frac{\nu}{\Lambda} - \sum_{\ell=1}^{\infty} \frac{h_\ell}{2^{1-\ell} \ell} \frac{1}{\nu^{2\ell}}, \quad h_0 = -\beta h_m. \quad (5.12)$$

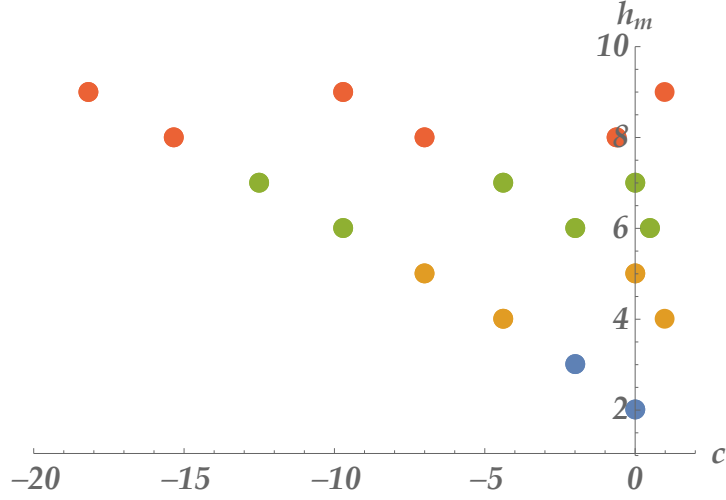


Figure 1. Values of (c, h_m) for 1, 2, 3, and 4 poles points (blue, orange, green, and red).

Then, (5.11) may be written in the equivalent form

$$\frac{\partial h_\ell}{\partial E_2} = \frac{\ell}{12} \sum_{n=0}^{\ell-1} h_n h_{\ell-1-n} + \beta \frac{\ell(2\ell-1)}{12} h_{\ell-1}. \quad (5.13)$$

Just to give a simple example, let us consider a 1-pole function and write the ν -dependent part of \tilde{F} as

$$\tilde{F}(\nu) = h_0 \log \frac{\nu}{\Lambda} - \beta \log \left(1 + \frac{\gamma_2 E_2}{\nu^2} \right). \quad (5.14)$$

Imposing (5.11) or (5.13), we recover the expression (5.7) for γ_2 . Besides, we also get the following constraint between α and β (excluding trivial solutions with constant prepotential)

$$4\alpha^2 - \beta^2 + 6\beta - 1 = 0, \quad \text{or} \quad 4\alpha^2 - \beta^2 + 10\beta - 1 = 0. \quad (5.15)$$

The condition (5.15) is indeed satisfied by the values in the first table of (5.9). However, there are infinite other pairs (α, β) that make (5.14) a solution of (5.11) that is not realized in the gauge theory. Of course, this is because (5.14) predicts all the higher order terms in the large ν expansion and this is correct in comparison with the actual Nekrasov formulas only for a finite set of values of α , and β . Actually, the admissibility of the Ansatz (5.3) is definitely non trivial. Anyway, we checked the validity of (5.11) for all the solutions we have presented.

As a final comment, we notice that using the explicit expressions for $\gamma_{2,2}$ and γ_2 it turns out that the constraint (5.15), up to an multiplicative factor, can be expressed as the ratio $\gamma_{2,2}/\gamma_2$:

$$\frac{\gamma_{2,2}}{\gamma_2} \sim (-4\alpha^2 + (\beta - 10)\beta + 1) (-4\alpha^2 + (\beta - 6)\beta + 1) = 0. \quad (5.16)$$

$\gamma_{2,2} = 0$ is of course a necessary condition in the Ansatz (5.3) to truncate the sum at $N = 1$. Looking at higher values of N , one gets similar constraints for any N . In fact, since the modular anomaly equation controls the dependence of \tilde{F} on E_2 , the consistency of our Ansatz with (5.11) imposes relations between all the coefficients of the form $\gamma_{2,X}$ with γ_X . For example, in the 2-poles case we have that

$$\frac{\gamma_{2,2,2}}{\gamma_{2,2}} \sim \frac{\gamma_{2,4}}{\gamma_4} \sim (-4\alpha^2 + (\beta - 18)\beta + 1) (-4\alpha^2 + (\beta - 14)\beta + 1), \quad (5.17)$$

and the constraint for α and β ensuring that $\gamma_{2,2,2} = \gamma_{2,4} = 0$ is

$$-4\alpha^2 + \beta^2 - 18\beta + 1 = 0 \quad \text{or} \quad -4\alpha^2 + \beta^2 - 14\beta + 1 = 0. \quad (5.18)$$

Similarly for 3-poles we get

$$\frac{\gamma_{2,2,2,2}}{\gamma_{2,2,2}} \sim \frac{\gamma_{2,2,4}}{\gamma_{2,4}} \sim \frac{\gamma_{2,6}}{\gamma_6} \sim (-4\alpha^2 + (\beta - 26)\beta + 1) (-4\alpha^2 + (\beta - 22)\beta + 1), \quad (5.19)$$

and so on.

5.2 A worked out 3-pole example

To appreciate the result of our analysis, let us consider in some details the first line of (A.1). The instanton partition function for $X = (\alpha, \beta) = (\frac{11}{2}, -4)$ is up to 12 instantons

$$\begin{aligned} \tilde{Z}_X^{\text{inst}}(\nu) = & 1 - \frac{12(\nu^2 - 37)q^2}{\nu^2 - 9} + \frac{54(\nu^4 - 69\nu^2 + 820)q^4}{(\nu^2 - 9)(\nu^2 - 4)} \\ & - \frac{8(11\nu^6 - 1078\nu^4 + 28679\nu^2 - 103212)q^6}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} - \frac{9(11\nu^6 - 1386\nu^4 + 3339\nu^2 + 239956)q^8}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} \\ & + \frac{540(\nu^6 - 154\nu^4 + 4333\nu^2 - 16948)q^{10}}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} - \frac{2(209\nu^6 - 38038\nu^4 + 1558361\nu^2 - 15934932)q^{12}}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} \\ & - \frac{648(\nu^6 - 210\nu^4 + 11109\nu^2 - 61860)q^{14}}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} + \frac{54(11\nu^6 - 2618\nu^4 + 225659\nu^2 - 2870732)q^{16}}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} \\ & + \frac{4(209\nu^6 - 55594\nu^4 + 3095981\nu^2 - 36365076)q^{18}}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} + \frac{96(11\nu^6 - 3234\nu^4 + 58779\nu^2 + 2749204)q^{20}}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} \\ & - \frac{216(19\nu^6 - 6118\nu^4 + 358351\nu^2 - 3592972)q^{22}}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} - \frac{(209\nu^6 - 73150\nu^4 + 18811121\nu^2 - 150221700)q^{24}}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)} + \dots \end{aligned} \quad (5.21)$$

This is far more involved than (2.10) and a brute force guess would not be possible. Nevertheless, it is a straightforward calculation to check that this is the expansion of

$$\tilde{Z}_X^{\text{inst}}(\nu) = [q^{-\frac{1}{12}}\eta(\tau)]^{12} \frac{\nu^6 - 14E_2\nu^4 + (\frac{140}{3}E_2^2 + \frac{7}{3}E_4)\nu^2 - \frac{280}{9}E_2^3 - \frac{14}{3}E_2E_4 - \frac{2}{9}E_6}{(\nu^2 - 9)(\nu^2 - 4)(\nu^2 - 1)}, \quad (5.22)$$

in agreement with the data in (A.1). The perturbative part of the prepotential is computed from (4.1) by expanding at large a . This gives, in the ν variable

$$\tilde{F}_X^{\text{pert}}(\nu) = 28 \log \frac{\nu}{\Lambda} - \frac{56}{\nu^2} - \frac{196}{\nu^4} - \frac{3176}{3\nu^6} - \frac{6818}{\nu^8} - \frac{240296}{5\nu^{10}} - \frac{1071076}{3\nu^{12}} - \frac{2742488}{\nu^{14}} + \dots, \quad (5.23)$$

and this is indeed the large ν expansion of

$$\tilde{F}_X^{\text{pert}}(\nu) = 28 \log \frac{\nu}{\Lambda} + 4 \log \left[\left(1 - \frac{1}{\nu^2}\right) \left(1 - \frac{4}{\nu^2}\right) \left(1 - \frac{9}{\nu^2}\right) \right], \quad (5.24)$$

where $28 = -\beta h_m = -(-4) \times 7$. Hence, the full quantum prepotential is in this case

$$\tilde{F} = 28 \log \frac{\nu}{\Lambda} + 4 \log \left[1 - \frac{14}{\nu^2} E_2 + \frac{7}{3\nu^4} (20 E_2^2 + E_4) - \frac{2}{9\nu^6} (140 E_2^3 + 21 E_2 E_4 + E_6) \right], \quad (5.25)$$

and one can check that (5.11) is satisfied.

6 Predictions for the torus 1-block

We have already seen that AGT implies explicit expressions for special torus blocks that we write stripping off the large h dominant term

$$\mathcal{F}_{h_m}^h(q, c) = \frac{q^{\frac{1}{12}}}{\eta(\tau)} \mathcal{H}_{h_m}^h(q, c). \quad (6.1)$$

From the 1-pole partition functions, we have obtained

$$\mathcal{H}_2^h(q, 0) = 1 + \frac{1 - E_2}{24h}, \quad \mathcal{H}_3^h(q, -2) = 1 + \frac{1 - E_2}{8h}. \quad (6.2)$$

From the 2-poles partition functions, we get similar expressions

$$\begin{aligned} \mathcal{H}_5^h(q, -7) &= 1 + \frac{15E_2^2 + E_4 - 80E_2(3h + 1) + 16(15h + 4)}{144h(4h + 1)}, \\ \mathcal{H}_4^h(q, -\frac{22}{5}) &= 1 + \frac{25E_2^2 + 2E_4 - 30E_2(40h + 9) + 3(400h + 81)}{960h(5h + 1)}, \\ \mathcal{H}_4^h(q, 1) &= 1 + \frac{E_2^2 - E_4 - 48E_2h + 48h}{48h(4h - 1)}, \\ \mathcal{H}_5^h(q, 0) &= 1 + \frac{3(5E_2^2 - 2E_4) - 10E_2(24h + 1) + 240h + 1}{192h(3h - 1)}. \end{aligned} \quad (6.3)$$

Notice that there are multiple entries in the tables (5.9) and (A.1) with the same value of (c, h_m) , but different (α, β) . Nevertheless, the associated block is consistently the same as soon as ν is expressed in terms of h . From the 3-poles partition functions, we get

$$\begin{aligned} \mathcal{H}_7^h(q, -\frac{25}{2}) &= 1 + \frac{1}{1152h(2h + 1)(16h + 5)} \left[-2(140E_2^3 + 21E_4E_2 + E_6) + 9(3584h^2 + 3248h + 729) \right. \\ &\quad \left. - 126E_2(16h + 9)^2 + 21(20E_2^2 + E_4)(16h + 9) \right], \\ \mathcal{H}_6^h(q, -\frac{68}{7}) &= 1 + \frac{1}{32256h(7h + 2)(7h + 3)} \left[-1715E_2^3 - 294E_4E_2 - 16E_6 \right. \\ &\quad \left. + 9(109760h^2 + 83496h + 15625) - 315E_2(56h + 25)^2 + 63(35E_2^2 + 2E_4)(56h + 25) \right], \end{aligned}$$

$$\begin{aligned}
\mathcal{H}_7^h(q, -\frac{22}{5}) &= 1 + \frac{1}{23040 h(5h-2)(5h+1)} \left[-5(875 E_2^3 - 294 E_4 E_2 - 176 E_6) \right. \\
&\quad \left. + 9(56000 h^2 + 14840 h + 729) - 315 E_2(40 h + 9)^2 + 21(125 E_2^2 - 14 E_4)(40 h + 9) \right], \\
\mathcal{H}_6^h(q, -2) &= 1 + \frac{1}{72 h(8h-3)(8h+1)} \left[-5 E_2^3 + 3 E_4 E_2 + 2 E_6 - 45 E_2(8 h + 1)^2 \right. \\
&\quad \left. + 9(5 E_2^2 - E_4)(8 h + 1) + 9(8 h + 1)(40 h + 1) \right], \\
\mathcal{H}_6^h(q, \frac{1}{2}) &= 1 + \frac{1}{3456 h(2h-1)(16h-1)} \left[-2(60 E_2^3 - 51 E_4 E_2 + 41 E_6) + 69120 h^2 \right. \\
&\quad \left. - 30 E_2(48 h + 1)^2 + 3(60 E_2^2 - 17 E_4)(48 h + 1) - 3312 h + 1 \right], \\
\mathcal{H}_7^h(q, 0) &= 1 + \frac{1}{216 h(8h-5)(8h-1)} \left[-105 E_2^3 + 63 E_4 E_2 - 22 E_6 + 12096 h^2 \right. \\
&\quad \left. - 21 E_2(24 h + 1)^2 + 21(5 E_2^2 - E_4)(24 h + 1) - 1008 h + 1 \right]. \tag{6.4}
\end{aligned}$$

All these expressions can be re-expanded at small q and compared with the recursion relations for the toroidal block with full agreement. It would be interesting to understand why such closed expressions are obtained at the special (c, h_m) points, for instance, by extending the results of [77, 78].

Acknowledgments

We thank A. Zein Assi for important comments and D. Fioravanti for clarifying discussions.

A 3- and 4-poles Nekrasov functions data

The 3-poles Nekrasov functions are

α	β	c	h_m	γ_2	γ_4	$\gamma_{2,2}$	γ_6	$\gamma_{2,4}$	$\gamma_{2,2,2}$	ν_1	ν_2	ν_3
$\frac{11}{2}$	-4	$-\frac{25}{2}$	7	-14	$\frac{7}{3}$	$\frac{140}{3}$	$-\frac{2}{9}$	$-\frac{14}{3}$	$-\frac{280}{9}$	1	2	3
$\frac{11}{8}$	$-\frac{1}{4}$	$-\frac{25}{2}$	7	$-\frac{7}{8}$	$\frac{7}{768}$	$\frac{35}{192}$	$-\frac{1}{18432}$	$-\frac{7}{6144}$	$-\frac{35}{4608}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
$\frac{19}{4}$	$-\frac{7}{2}$	$-\frac{68}{7}$	6	$-\frac{35}{4}$	$\frac{7}{8}$	$\frac{245}{16}$	$-\frac{1}{36}$	$-\frac{49}{96}$	$-\frac{1715}{576}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\frac{19}{14}$	$-\frac{2}{7}$	$-\frac{68}{7}$	6	$-\frac{5}{7}$	$\frac{2}{343}$	$\frac{5}{49}$	$-\frac{16}{1058841}$	$-\frac{2}{7203}$	$-\frac{5}{3087}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{5}{7}$
$\frac{17}{4}$	$-\frac{5}{2}$	$-\frac{22}{5}$	7	$-\frac{35}{4}$	$-\frac{49}{24}$	$\frac{875}{48}$	$\frac{55}{36}$	$\frac{245}{96}$	$-\frac{4375}{576}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$\frac{17}{10}$	$-\frac{2}{5}$	$-\frac{22}{5}$	7	$-\frac{7}{5}$	$-\frac{98}{1875}$	$\frac{7}{15}$	$\frac{176}{28125}$	$\frac{98}{9375}$	$-\frac{7}{225}$	$\frac{1}{5}$	$\frac{3}{5}$	1
$\frac{7}{2}$	-2	-2	6	-5	-1	5	$\frac{2}{9}$	$\frac{1}{3}$	$-\frac{5}{9}$	0	1	2
$\frac{7}{4}$	$-\frac{1}{2}$	-2	6	$-\frac{5}{4}$	$-\frac{1}{16}$	$\frac{5}{16}$	$\frac{1}{288}$	$\frac{1}{192}$	$-\frac{5}{576}$	0	$\frac{1}{2}$	1
$\frac{17}{6}$	$-\frac{4}{3}$	$\frac{1}{2}$	6	$-\frac{10}{3}$	$-\frac{17}{27}$	$\frac{20}{9}$	$-\frac{82}{729}$	$\frac{34}{243}$	$-\frac{40}{243}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{3}$
$\frac{17}{8}$	$-\frac{3}{4}$	$\frac{1}{2}$	6	$-\frac{15}{8}$	$-\frac{51}{256}$	$\frac{45}{64}$	$-\frac{41}{2048}$	$\frac{51}{2048}$	$-\frac{15}{512}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{4}$
$\frac{13}{4}$	$-\frac{3}{2}$	0	7	$-\frac{21}{4}$	$-\frac{21}{16}$	$\frac{105}{16}$	$-\frac{11}{32}$	$\frac{63}{64}$	$-\frac{105}{64}$	$\frac{1}{2}$	1	2
$\frac{13}{6}$	$-\frac{2}{3}$	0	7	$-\frac{7}{3}$	$-\frac{7}{27}$	$\frac{35}{27}$	$-\frac{22}{729}$	$\frac{7}{81}$	$-\frac{35}{243}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$

(A.1)

The 4-poles data is split in the following two tables. The first contains the values of (c, h_m) and the poles positions for each pair (α, β) .

α	β	c	h_m	ν_1	ν_2	ν_3	ν_4
7	-5	$-\frac{91}{5}$	9	1	2	3	4
$\frac{7}{5}$	$-\frac{1}{5}$	$-\frac{91}{5}$	9	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
$\frac{25}{4}$	$-\frac{9}{2}$	$-\frac{46}{3}$	8	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
$\frac{25}{18}$	$-\frac{2}{9}$	$-\frac{46}{3}$	8	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	$\frac{7}{9}$
$\frac{23}{4}$	$-\frac{7}{2}$	$-\frac{68}{7}$	9	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$
$\frac{23}{14}$	$-\frac{2}{7}$	$-\frac{68}{7}$	9	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{5}{7}$	1
5	-3	-7	8	0	1	2	3
$\frac{5}{3}$	$-\frac{1}{3}$	-7	8	0	$\frac{1}{3}$	$\frac{2}{3}$	1
$\frac{11}{3}$	$-\frac{5}{3}$	$-\frac{3}{5}$	8	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{7}{3}$
$\frac{11}{5}$	$-\frac{3}{5}$	$-\frac{3}{5}$	8	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{7}{5}$
3	-1	1	9	0	1	1	2

(A.2)

The next table lists the values of the γ -coefficients.

α	β	γ_2	γ_4	$\gamma_{2,2}$	γ_6	$\gamma_{2,4}$	$\gamma_{2,2,2}$	$\gamma_{4,4}$	$\gamma_{2,6}$	$\gamma_{2,2,4}$	$\gamma_{2,2,2,2}$
7	-5	-30	$\frac{21}{2}$	$\frac{525}{2}$	$-\frac{10}{3}$	$-\frac{175}{2}$	$-\frac{4375}{6}$	$\frac{41}{16}$	$\frac{25}{3}$	$\frac{875}{8}$	$\frac{21875}{48}$
$\frac{7}{5}$	$-\frac{1}{5}$	$-\frac{6}{5}$	$\frac{21}{1250}$	$\frac{21}{50}$	$-\frac{2}{9375}$	$-\frac{7}{1250}$	$-\frac{7}{150}$	$\frac{41}{6250000}$	$\frac{1}{46875}$	$\frac{7}{25000}$	$\frac{7}{6000}$
$\frac{25}{4}$	$-\frac{9}{2}$	-21	$\frac{21}{4}$	$\frac{945}{8}$	-1	$-\frac{189}{8}$	$-\frac{2835}{16}$	$\frac{15}{64}$	$\frac{3}{4}$	$\frac{567}{64}$	$\frac{8505}{256}$
$\frac{25}{18}$	$-\frac{2}{9}$	$-\frac{28}{27}$	$\frac{28}{2187}$	$\frac{70}{243}$	$-\frac{64}{531441}$	$-\frac{56}{19683}$	$-\frac{140}{6561}$	$\frac{20}{14348907}$	$\frac{64}{14348907}$	$\frac{28}{531441}$	$\frac{35}{177147}$
$\frac{23}{4}$	$-\frac{7}{2}$	-21	$-\frac{21}{4}$	$\frac{1029}{8}$	$\frac{53}{3}$	$\frac{245}{8}$	$-\frac{12005}{48}$	$-\frac{553}{64}$	$-\frac{371}{12}$	$-\frac{1715}{64}$	$\frac{84035}{768}$
$\frac{23}{14}$	$-\frac{2}{7}$	$-\frac{12}{7}$	$-\frac{12}{343}$	$\frac{6}{7}$	$\frac{3392}{352947}$	$\frac{40}{2401}$	$-\frac{20}{147}$	$-\frac{316}{823543}$	$-\frac{3392}{2470629}$	$-\frac{20}{16807}$	$\frac{5}{1029}$
5	-3	-14	$-\frac{7}{2}$	$\frac{105}{2}$	6	$\frac{21}{2}$	$-\frac{105}{2}$	$-\frac{15}{16}$	-3	$-\frac{21}{8}$	$\frac{105}{16}$
$\frac{5}{3}$	$-\frac{1}{3}$	$-\frac{14}{9}$	$-\frac{7}{162}$	$\frac{35}{54}$	$\frac{2}{243}$	$\frac{7}{486}$	$-\frac{35}{486}$	$-\frac{5}{34992}$	$-\frac{1}{2187}$	$-\frac{7}{17496}$	$\frac{35}{34992}$
$\frac{11}{3}$	$-\frac{5}{3}$	$-\frac{70}{9}$	$-\frac{133}{54}$	$\frac{875}{54}$	$-\frac{610}{729}$	$\frac{665}{162}$	$-\frac{4375}{486}$	$\frac{6667}{34992}$	$\frac{1525}{6561}$	$-\frac{3325}{5832}$	$\frac{21875}{34992}$
$\frac{11}{5}$	$-\frac{3}{5}$	$-\frac{14}{5}$	$-\frac{399}{1250}$	$\frac{21}{10}$	$-\frac{122}{3125}$	$\frac{1197}{6250}$	$-\frac{21}{50}$	$\frac{20001}{6250000}$	$\frac{61}{15625}$	$-\frac{1197}{125000}$	$\frac{21}{2000}$
3	-1	-6	$-\frac{3}{2}$	$\frac{21}{2}$	$-\frac{2}{3}$	$\frac{5}{2}$	$-\frac{35}{6}$	$-\frac{7}{16}$	$\frac{1}{3}$	$-\frac{5}{8}$	$\frac{35}{48}$

(A.3)

B On the structure of higher instanton Nekrasov functions

From the analysis in the main text, it is clear that in all the cases the k -instanton contribution to $\tilde{F}^{\text{inst}}(\nu)$ is a rational function of ν : in particular in the 1-pole cases the structure of the $\tilde{F}_{X_1,k}^{\text{inst}}(\nu)$ is in close analogy with the Nekrasov-Shatashvili limit studied in [63]. It can be instructive to repeat some of the steps of the analysis of [63] and highlight various differences. Taking the 1-pole point X_1 case as example, we have

$$\tilde{F}_{X_1,k}^{\text{inst}}(\nu) = \frac{P_k(\nu)}{(\nu^2 - 1)^k}, \quad (\text{B.1})$$

where $P_k(\nu)$ are even polynomials of degree $2k$. The first cases are explicitly

$$\begin{aligned} P_1(\nu) &= -8(\nu^2 - 7), \\ P_2(\nu) &= -12(\nu^4 - 14\nu^2 + 61), \\ P_3(\nu) &= -\frac{32}{3}(\nu^6 - 21\nu^4 + 363\nu^2 - 1207), \\ P_4(\nu) &= -2(7\nu^8 - 196\nu^6 + 5442\nu^4 - 51796\nu^2 + 129487), \\ P_5(\nu) &= -\frac{48}{5}(\nu^{10} - 35\nu^8 + 2410\nu^6 - 44470\nu^4 + 289205\nu^2 - 578887), \\ P_6(\nu) &= -16(\nu^{12} - 42\nu^{10} + 2715\nu^8 - 79580\nu^6 + 948615\nu^4 - 4655130\nu^2 + 7764733), \\ &\dots \end{aligned} \quad (\text{B.2})$$

In the cases treated in [63], the expansion around the poles revealed a *selection rule* forbidding even poles, *i.e.* the functions $\tilde{F}_{\text{NS},k}^{\text{inst}}(\nu)$ have always the form

$$\tilde{F}_{\text{NS},k}^{\text{inst}}(\nu) = \frac{d_1^{(k)}}{(\nu - 1)^{2k-1}} + \frac{d_2^{(k)}}{(\nu - 1)^{2k-3}} + \dots + \frac{d_k^{(k)}}{(\nu - 1)} + \text{regular}. \quad (\text{B.3})$$

While the structure is extremely similar, the *selection rule* is no longer true here. At the point X_1 , the Laurent expansion of $\tilde{F}_{X_1,k}^{\text{inst}}(\nu)$ around $\nu = 1$ has the generic form

$$\tilde{F}_{X_1,k}^{\text{inst}}(\nu) = \frac{d_0^{(k)}}{(\nu-1)^k} + \frac{d_1^{(k)}}{(\nu-1)^{k-1}} + \cdots + \frac{d_{k-1}^{(k)}}{(\nu-1)} + \text{regular}, \quad (\text{B.4})$$

where all the coefficients are non vanishing. The same is true for all the other three 1-pole cases X_2, X_3, X_4 . As in [63], we can rewrite the k -instanton functions in terms of the $d_p^{(k)}$ in the exact form

$$\tilde{F}_{X_1,k}^{\text{inst}}(\nu) = c_k + \sum_{p=0}^{k-1} d_p^{(k)} \left(\frac{(-1)^{k-p}}{(\nu-1)^{k-p}} + \frac{1}{(\nu+1)^{k-p}} \right). \quad (\text{B.5})$$

The coefficients c_k capture the ν independent part (*i.e.* the term proportional to the logarithms of the Dedekind function). The coefficients $d_p^{(k)}$ can be obtained from the expansion of the exact expressions of $\tilde{F}_{X_i,k}^{\text{inst}}(\nu)$. For X_1 they read

$$\begin{aligned} d_0^{(k)} &= 2 \frac{12^k}{k}, \\ d_1^{(k)} &= d_0^{(k)} \times \frac{3k}{4}, \\ d_2^{(k)} &= d_0^{(k)} \times \frac{1}{288} k(81k-19), \\ d_3^{(k)} &= d_0^{(k)} \times \frac{k(243k^2-171k+470)}{3456}, \\ d_4^{(k)} &= d_0^{(k)} \times \frac{k(2187k^3-3078k^2+17281k-9662)}{165888}, \\ &\dots \end{aligned} \quad (\text{B.6})$$

and so on.

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